Quantum Computing

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Blurb for the back of the book:

Quantum computing promises a fundamental transformation across science and industry. This accessible book provides a comprehensive introduction, guiding students and practitioners into this rapidly evolving field with no physics prerequisites.

The book's novel starting point is marble physics. Readers begin with ordinary marbles moving along a graph, then watch superposition, interference, and entanglement emerge from one small change to the rules. From this intuitive foundation, the text progresses step-by-step through quantum mechanics, quantum gates and circuits, error correction, cryptography, and the great quantum algorithms. Every major algorithm is fully worked out with concrete numbers, including Shor's algorithm factoring an actual integer, and the coverage extends to the research frontier, from quantum complexity theory to recent breakthroughs.

The book transforms abstract ideas into hands-on practice, featuring detailed examples, pedagogical exercises, and programming assignments in Qiskit, the widely adopted open-source platform that lets students run their circuits on real quantum computers. Every section is accompanied by an online video in which the author walks through the material, making this an ideal text for undergraduate and graduate courses, self-study, and professional development.

Three sentence blurb:

This book offers a rigorous yet highly accessible introduction to quantum computing, beginning with an intuitive marble model from which superposition, interference, and entanglement emerge before the reader's eyes. Every major algorithm is completely worked out with concrete examples, and abundant exercises and Qiskit programming assignments let students run their own circuits on real quantum computers. With coverage reaching from mathematical foundations to the research frontier, and an accompanying video for every section, the text integrates deep theory with modern hands-on practice.

The Cambridge University Press would like to thank the anonymous peer reviewers who provided comments on drafts of this book. The generous work of academic experts is essential for establishing the authority and quality of our publications. We acknowledge with gratitude the contributions of these otherwise uncredited readers.

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In memory of
Alex Heller

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Preface

Quantum computing is a thrilling, fast-developing field that harnesses the strange and powerful phenomena of quantum mechanics to push computers beyond their classical limits. These machines promise to revolutionize the digital world. News outlets report daily on the race to build large-scale quantum computers and achieve quantum advantage, while engineers and scientists push the boundaries of what is possible.

This book is not only about tomorrow's technology. It is also about deep ideas that shape our world. For a long time, there has existed a very large, robust quantum computer: our universe. This quantum computer operates under the laws of quantum mechanics with astonishing efficiency. To study quantum computing is to study the very fabric of reality and the awestruck observers it has produced. At its core, quantum mechanics is built on experiments that reveal profound truths about the world we inhabit. This book explores not only quantum technology but also quantum information, which is the dialogue between the universe and its observers.

So why should you read this particular book? Because it was written to prove that quantum computing's fearsome reputation is undeserved. People will tell you that you need years of physics before you can even begin. You do not. If you know some basic mathematics and you are willing to think, you can understand quantum computing. All of it.

The heart of the book is a simple idea. We begin with marbles moving along the edges of a graph. Everyone can picture that. We then ask what happens if the marbles move probabilistically. Finally we make one small change to the rules and the marbles become quantum. With that one change, superposition, interference, and entanglement appear before your eyes. You do not have to take quantum mechanics on faith. You watch it emerge from systems you already understand.

The rest of the book keeps that promise. This text avoids hand-waving. Almost every idea is explained at its deepest level and is supported by worked examples. Every major algorithm is described completely and then worked out with a small concrete example. You will watch Shor's algorithm factor an actual number, step by step, with nothing hidden. Exercises throughout the book build the skills needed to go further, while programming assignments encourage students to experiment. We begin with simple operations on complex numbers and progress to full-scale quantum computer simulators. Readers also get in on the excitement by programming real quantum computers using Qiskit, a popular and freely available platform.

The book also travels far. From the basics to cutting-edge research, it gen-

tly guides students through the foundations of quantum computing and onward to quantum cryptography, quantum error correction, and quantum complexity theory. We end at the research frontier, where quantum computing meets black holes, spacetime, and the limits of what can be known. Along the way we meet algorithms discovered only in the last few years. The goal is to provide a deep understanding of both the essence and the details of the field. We go "under the hood" to see how these fundamentally different machines work. True understanding comes from dealing with the math and physics that underlie quantum computing. By the end, students will be equipped to read research papers and contribute to the ongoing quantum revolution.

And you will not be alone on this trip. Every section of the book has an accompanying video in which I walk through the material with you. Read a section, watch the video, do the exercises, and the ideas will be yours.

Quantum computing is one of the great scientific adventures of our time. This book was written so that you can join it. Welcome aboard.

Organization

Quantum computation has emerged as a central area at the intersection of computer science, physics, electrical engineering, and mathematics. Its study requires a blend of physical intuition, mathematical structure, and computational thinking, and it offers new ways of understanding information and the natural world. This book is designed to provide a clear and rigorous introduction to the field, accessible to undergraduates, graduate students, and self-learners with backgrounds in any of these disciplines.

The book is organized to ease the students into the world of quantum computing. The first part of the book, Chapters 1–4, contains preliminaries that help students slowly get their toes wet. This includes some history, mathematical background, intuitions, and a gentle introduction to the fundamentals of quantum mechanics.

In the second part of the book, Chapters 5–8, we start swimming the warm waters of quantum computing. This part of the text is about the quantum bits and quantum gates that make up quantum computers. We use these concepts to describe error correction procedures, cryptography protocols, and the exciting world of quantum algorithms.

The final part of the book, Chapters 9–12, delves into advanced topics. The book describes quantum computing from a theoretical computer science point of view. We also see how physical quantum computers are actually built. The basics of quantum information theory are explored. At the end, we present some ideas on the foundations of quantum mechanics. These topics will take us far from the shallows.

In detail, the chapters cover the following topics.

- **Chapter 1: Introduction** places quantum computing into context. It is a

historical introduction to the entire field. The chapter starts with a short history of some aspects of quantum theory and some history of computing. With this in hand, the chapter goes on to discuss the history of quantum computing.

- **Chapter 2: Mathematical Preliminaries** describes the mathematics that will be needed for the rest of the text. With every mathematical concept or structure, the chapter foreshadows how that idea or construct is used in quantum mechanics and quantum computing.
- **Chapter 3: Marble Physics** is a novel, motivating chapter that uses toy marbles as an intuitive introduction to the central ideas and experiments of quantum mechanics and quantum computing.
- **Chapter 4: Quantum Mechanics** formally describes the laws of quantum mechanics. With all the foreshadowing done earlier, these ideas and structures will be like old familiar friends and will be easily understandable.
- **Chapter 5: The Building Blocks of Quantum Computing** describes the quantum analogies of bits, gates, and circuits. This is the basic architecture of quantum computing. There is also a section on some of the limitations quantum computing has when dealing with quantum information, including the no-cloning theorem. With this knowledge in hand, the text introduces Qiskit so one can start programming a quantum computer.
- **Chapter 6: Error Correction** is the first application of quantum computing. The chapter shows how errors are dealt with in classical computers and quantum computers. We also show how error correction is believed to be a central aspect of our universe.
- **Chapter 7: Cryptography** describes how quantum computers can be used to communicate in secret. The text also touches on what cryptography might look like when actual quantum computers destroy old cryptographic ways of communicating. The chapter closes with a discussion of some nifty protocols that deal with entanglement such as dense coding and teleportation.
- **Chapter 8: Algorithms** is the heart of the book. Here the text goes through many different quantum algorithms with applications in many areas. The chapter explains what problems the algorithms solve, how they work, and how they compare to classical algorithms.
- **Chapter 9: Theoretical Computing** is about classifying computational problems and establishing complexity hierarchies. The text describes what is known and unknown about the relationship between classical and quantum computing. The chapter ends with some ways that quantum complexity is used to describe the space-time structure of our universe.
- **Chapter 10: Implementing Quantum Computers** discusses several different ways that quantum computers are physically constructed. The chapter explains the different hardware models that are used in current quantum computers.

The following two supplementary chapters are freely available on my web page and at www.cambridge/XXX.

- **Chapter 11: Information Theory** is a field that is related to communication

and computing. We begin with some information theory about the classical world and then we move on to the world of quantum information theory.
- **Chapter 12: Quantum Foundations** goes through different interpretations of quantum mechanics. Several important theories and experiments are explained using the tools and ideas that we developed in quantum computing.

This book is an interconnected whole. To stitch the narrative together, we have used Foreshadowing boxes throughout the text, where we hint at the fascinating ideas waiting in later sections and chapters. There are also Background Reminder boxes that point you back to ideas we have already encountered and that should be reviewed before going further. By connecting where you have been with where you are going, this textbook makes learning quantum computing easier.

For the most part, once students have completed Chapter 5 - "The Building Blocks of Quantum Computing," the rest of the chapters are independent of each other and can be read in any desired order.

At the end of each chapter there are bibliographical references where students can pursue further study.

For Instructors

Since quantum computing sits at the intersection of physics, mathematics, electrical engineering, and computer science, students approach this subject with widely varying backgrounds. Some may be fluent in linear algebra but unfamiliar with quantum mechanics, while others may have strong programming skills but limited exposure to computer architecture. This book is designed to accommodate that diversity by being a flexible resource. Rather than prescribing a single path through the material, it offers a framework that can be adapted to a variety of teaching styles and learning contexts, including some suggestions for tailoring the course to a class's needs.

Chapter Notes for Teaching

- **Chapter 1: Introduction.** This chapter is a historical introduction to the entire field. Almost every topic that will be met in the text is discussed in a non-technical form here. This chapter should be assigned for the student to read on their own.
- **Chapter 2: Mathematical Preliminaries.** Here we provide the essential mathematical background for the rest of the text. Students majoring in electrical engineering, physics, or mathematics may already be familiar with some of the material, while others may be encountering it for the first time. To tailor instruction effectively, instructors are encouraged to distribute an anonymous written survey covering the topics in Chapter 2. The results can help shape a course that meets students where they are. The course can then reinforce what is needed without overloading them with material they have already mastered. Because some students may feel overwhelmed by starting with a dense block of mathematics, it is strongly recommended that this

chapter be taught in an "as needed" manner. Sections 2.2–2.4 on linear algebra and Hilbert spaces are essential before beginning Chapter 4 on quantum mechanics and should be covered accordingly. Other sections can be introduced just prior to their application. The material on modular arithmetic and number theory should be taught before RSA encryption (Section 7.1) and Shor's algorithm (Section 8.8). Similarly, the section on probability and statistics is best covered before supplementary Chapter 11.

- **Chapter 4: Quantum Mechanics.** Students are introduced to the foundational principles of quantum mechanics. Students majoring in physics or electrical engineering may have encountered some of this material in prior coursework, which can be helpful. Instructors should be aware that many undergraduate quantum mechanics courses focus narrowly on solving Schrödinger's equation. As a result, students often lack exposure to key conceptual tools such as Hilbert spaces, superposition, and entanglement. These ideas are central to quantum computing and are presented here with clarity and precision. Even students with a physics background may benefit from revisiting these ideas in this context. For that reason, it is strongly recommended that instructors spend several class sessions guiding students through this chapter, ensuring that all learners (regardless of prior exposure) develop a solid conceptual foundation for the chapters that follow.

Suggested Course Paths

- **Electrical Engineering Course:** An undergraduate electrical engineering course can typically assume that students are familiar with complex numbers and linear algebra. A recommended path begins with Chapters 4 through 8, followed by Chapters 10 and 11. Programming exercises should be required, as they reinforce key algorithmic concepts and help bridge theory with practical implementation.
- **Physics Course:** Physics students often have a strong foundation in mathematical methods, including complex numbers and linear algebra. This course should cover Chapters 4, 5, 6, 8, 10, and 12. These chapters emphasize quantum mechanics, physical models of computation, and foundational questions that resonate with physics majors. The supplementary Chapter 12, in particular, offers philosophical depth that complements the technical material.
- **Mathematics Course:** For mathematics majors, the course can begin with Chapters 3 through 5, which introduce marble physics, quantum mechanics, and the mathematical structure of quantum systems. It can then proceed to Chapters 8, 9, and 11, focusing on algorithms, complexity theory, and information theory.
- **Computer Science Course:** A one-semester computer science course can effectively cover Chapters 2 through 9. This includes mathematical preliminaries, quantum gates, algorithms, and formal models. Programming exercises should be required, as they provide essential hands-on experience and deepen understanding of algorithmic design.
- **Interdisciplinary Majors Course:** This category includes science majors

Table 0.1: Summary of suggested courses. Legend: Read = Assigned as independent reading, Class = Covered during lectures, dash = chapter not used in that track.

	Elec. Eng.	Physics	Math	Comp. Sci.	Interd.
1 Intro	Read	Read	Read	Read	Read
2 Math Prelim.	Read	Read	–	Class	Class
3 Marble Physics	Read	–	Class	Class	Class
4 Quantum Mech	Class	Class	Class	Class	Class
5 Building Blocks	Class	Class	Class	Class	Class
6 Error Correc.	Class	Class	–	Class	Class
7 Cryptography	Class	–	–	Class	Class
8 Algorithms	Class	Class	Class	Class	–
9 Theory Comp.	–	Read	Class	Class	–
10 Implement.	Class	Class	–	–	–
11 Info Theory	Class	–	Class	–	–
12 Foundations	–	Class	Read	–	Class

from diverse fields, as well as students with philosophical or foundational interests in quantum computing. Chapters 2 through 7 should be taught in class to ensure conceptual grounding. The supplementary Chapter 12 is highly recommended for students interested in the philosophical implications of quantum mechanics and computation.

- **Second Semester Course:** Follow-up courses can explore advanced topics not covered in the first term. These may include deeper algorithmic techniques, cryptographic applications, or foundational issues in quantum theory. Instructors are encouraged to select chapters based on student interest and background.
- **Graduate Course:** Graduate-level courses can cover the entire text in one semester. Students should be encouraged to select advanced sections for presentation and discussion. This approach fosters deeper engagement with the material and allows for specialization based on research interests.

See Table 0.1 for a summary of these course paths.

Ancillaries

This text does not stand alone. I maintain a web page for the text at

`http://www.sci.brooklyn.cuny.edu/~noson/QComp.html`.

The web page contains the supplementary chapters, links to video lectures for every section of the book, links to relevant web pages and GitHub pages with solutions to

programming exercises, and much more. Associated with this book is a complete answer key with the solution to every exercise in the book

Acknowledgments

This book is dedicated to the memory of my thesis advisor, Alex Heller (July 9, 1925 – January 31, 2008).

Heller was a world-class mathematician whose career began as a teenager when he was part of the Manhattan Project at Los Alamos. He went on to earn his Ph.D. at Columbia University under the supervision of Samuel Eilenberg, followed by postdoctoral positions at the Institute for Advanced Study and Harvard University. From there he joined the faculty at the University of Illinois at Urbana–Champaign. In 1965, Heller returned to New York as a founding member of the mathematics Ph.D. program at the Graduate Center of the City University of New York (CUNY), where he was named Distinguished Professor in 1970 and remained until his retirement in 2002. He supervised 17 Ph.D. students, held visiting professorships in Oxford, Paris, Milan, and Sydney, and was the founding co-editor of the *Journal of Pure and Applied Algebra*. Independently of Alexander Grothendieck, he formulated and developed the theory of derivators, a deep mathematical framework that provides a unified language for algebraic geometry, algebraic topology, and representation theory.

Beyond his theorems, Alex was a true Renaissance man. His interests spanned cognitive science, philosophy, poetry, and cooking. He was a cultured gentleman, loved by all who knew him, and fondly remembered for his warm, welcoming smile and the large cigar that was his constant companion.

I first met Professor Heller in my second year of graduate school at the Graduate Center. He introduced me to the world of mathematical research and opened my eyes to his unique way of seeing mathematics. His knowledge was immense, but what was most striking was his clarity and his perspective. He always seemed to view mathematics from a higher vantage point with a kind of serene, panoramic understanding. He saw unity and a deep interconnectedness in mathematics. Our conversations ranged far beyond mathematics. We talked about philosophy, language, history, and politics, and he made every conversation feel like an invitation to think more deeply.

I received my Ph.D. in 1996, but our relationship did not end there. We continued to meet once or twice a week, right up until a few days before his passing. In a very real sense, he gave me a twelve-year postdoctoral research position. I owe him so much for the vision he tried to show me, for the intellectual independence he insisted on, for his kindness, and for his friendship. He will always be warmly remembered.

A word must be said about the mathematics department at the Graduate Center. Alongside Professor Heller, the department was home to a remarkable group of mathematicians who created an atmosphere that was both intellectually serious and genuinely welcoming. Sitting around the common room, one could always find stimulating conversation, sharp insight, and good humor. The professors who

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Over the years, I have had many teachers and friends who have accompanied me through the quantum realm. I really learned the foundations of quantum mechanics while working on a project with Adam Brandenburger Brandenburger and Yanofsky (2008). Karl Svozil is a good friend who is always an email away, and has saved me from a lot of quantum nonsense. Other guides, fellow travelers, and friends in the quantum realm include Michael Dascal, Edgar Feldman, Ari Halpert, Jonathan Hanon, Mark Hillery, Eliyahu Hershfeld, Peter Hines, Klaas Landsman, Avi Rabinowitz, Michael Vitz, Joel Zablow, and Mark Zelcer. I gained much from their wisdom and their helpful conversations.

Brooklyn College has been a large part of my life since I entered as a freshman in September of 1985. I am grateful for the intellectual environment that it provides. I thank President Michelle J. Anderson, Dean Peter Tolias, and Chairman Hui Chen who have all been very encouraging of my work. My colleagues and friends are always there for a chat and for help editing various drafts. In particular, I would like to thank David Arnow, Miriam Briskman, Hui Chen, Katherine Chuang, Eva Cogan, James Cox, Lawrence Goetz, Keith Harrow, Yedidyah Langsam, Moshe Lach, Ira Rudowsky, Rohit Parikh, Panneer Selvam Santhalingam, Bridget Sheridan, Alexander Sverdlov, Joseph Thurm, and Gerald Weiss.

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